PHYC 511
Spring 2019

Problem Session 8 03/20/2019

(1) Problem 7.28, Jackson.

(2) Problem 7.29, Jackson.

(1) Let us write:

$$\vec{E}(x,y,z,t) = \left[E_0(x,y)(\hat{e}_1 \pm i\hat{e}_2) + E_z(x,y)\,\hat{e}_3\right] e^{i(kz-\omega t)}$$

Then, $\vec{\nabla}\cdot\vec{E} = 0$ implies that:

$$\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y} + ikE_z = 0 \Rightarrow E_z = \frac{i}{k}\left(\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y}\right)$$

Thus:

$$\boxed{\vec{E}(x,y,z,t) = \left[E_0(x,y)(\hat{e}_1 \pm i\hat{e}_2) + \frac{i}{k}\left(\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y}\right)\hat{e}_3\right] e^{i(kz-\omega t)}}$$

To find $\vec{B}$, we note that:

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \underset{\substack{\uparrow \\ \text{for a harmonic} \\ \text{function of time}}}{i\omega}\vec{B} = \vec{\nabla}\times\vec{E} \Rightarrow \vec{B} = \frac{-i}{\omega}\vec{\nabla}\times\vec{E}$$

$$\vec{\nabla}\times\vec{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{e}_1 + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{e}_2 + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{e}_3]$$

We can neglect $\frac{\partial E_z}{\partial y}$ and $\frac{\partial E_z}{\partial x}$ since the amplitude modulation is slowly varying. Then:

$$\vec{\nabla}\times\vec{E} \simeq \left[\mp kE_0\hat{e}_1 - ikE_0\hat{e}_2 + \left(\pm i\frac{\partial E_0}{\partial x} - \frac{\partial E_0}{\partial y}\right)\right] e^{i(kz-\omega t)} \Rightarrow$$

$$\vec{B} = \frac{-i}{\omega}\left[\pm k\, E_0\, \hat{e}_1 - i k E_0 \hat{e}_2 \pm k \times \frac{i}{k}\left(\frac{\partial E_0}{\partial x} \mp i \frac{\partial E_0}{\partial y}\right)\hat{e}_3\right] \Rightarrow$$

$$\vec{B} = \mp \frac{ik}{\omega}\left[E_0 \hat{e}_1 \pm i E_0 \hat{e}_2 + \frac{i}{k}\left(\frac{\partial E_0}{\partial x} - \frac{\partial E_0}{\partial y}\right)\hat{e}_3\right]$$

We note that $\frac{k}{\omega} = \frac{1}{\frac{v}{n}} = \sqrt{\nu\epsilon}$. Also, the term inside the bracket is just $\vec{E}$. Hence;

$$\boxed{\vec{B} = \mp i\sqrt{\nu\epsilon}\;\vec{E}}$$

(2) We have:

$$\langle J_z \rangle = \frac{1}{2}\varepsilon \, \mathrm{Re} \int \vec{x} \times (\vec{E} \times \vec{B}^*) d^3x \cdot \hat{e}_3 = \frac{1}{2}\varepsilon \int \vec{x} \times \mathrm{Re}(\vec{E} \times \vec{B}^*) \, d^3x \cdot \hat{e}_3$$

time-averaged value

Since $B = \mp i\sqrt{\mu\varepsilon}\,\vec{E}$, then:

$$\langle J_z \rangle = \pm \frac{1}{2} \frac{\varepsilon}{k} \sqrt{\mu\varepsilon} \int \vec{x} \times \mathrm{Re}(\vec{E} \times i\vec{E}^*) \, d^3x$$

$\vec{E} \times i\vec{E}^*$ is real as $(\vec{E} \times i\vec{E}^*)^* = -i(\vec{E}^* \times \vec{E}) = i(\vec{E} \times \vec{E}^*)$. Substituting for $\vec{E}$ from the previous problem, and calculating $\langle J_z \rangle$, we find:

$$\langle J_z \rangle = \pm \frac{\varepsilon}{k} \sqrt{\mu\varepsilon} \int \left(-x E_0 \frac{\partial E_0}{\partial x} - y E_0 \frac{\partial E_0}{\partial y}\right) d^3x$$

But:

$$\int \left(-x E_0 \frac{\partial E_0}{\partial x} - y E_0 \frac{\partial E_0}{\partial y}\right) d^3x = \int \left(-\frac{1}{2} x \frac{\partial E_0^2}{\partial x} - \frac{1}{2} y \frac{\partial E_0^2}{\partial x}\right) d^3x =$$

$$\int \left[-\frac{1}{2}\frac{\partial}{\partial x}(x E_0^2) - \frac{1}{2}\frac{\partial}{\partial y}(y E_0^2) + \frac{1}{2}E_0^2 + \frac{1}{2}E_0^2\right) d^3x = \int E_0^2 \, d^3x$$

Volume integral of total derivatives vanish due to finite extension of $\vec{E}$ in the x and y directions

Therefore:

$$\langle J_z \rangle = \pm \frac{\varepsilon}{k} \sqrt{\mu\varepsilon} \int E_0^2 \, d^3x$$

(5)

On the other hand:

$$\langle U \rangle = \int \left(\frac{1}{2} \varepsilon E_0^2 + \frac{1}{2} \frac{B_0^2}{\mu} \right) d^3x = \varepsilon \int E_0^2 d^3x$$

In vacuum, we have $\sqrt{\mu\varepsilon} = \frac{1}{c}$ and $\omega = ck$. Hence:

$$\boxed{\frac{\langle J_z \rangle}{\langle U \rangle} = \pm \frac{\sqrt{\mu\varepsilon}}{k} = \pm \frac{1}{\omega}}$$

The interpretation of this result in terms of photons is clear. The $\pm$ helicity states of a photon carry a spin $\pm\hbar$ in the direction of propagation, while the energy of the photon is $\hbar\omega$. Therefore, for a single photon, we have $\frac{\langle J_z \rangle}{\langle E \rangle} = \pm \frac{1}{\omega}$. The relation holds for the electromagnetic wave, which can be considered as a collection of photons.

As for $\langle J_x \rangle$ and $\langle J_y \rangle$, we have:

$$\langle J_x \rangle = \mp \frac{\varepsilon}{k} \sqrt{\mu\varepsilon} \int \left(E_0^2 k y + E_0 \frac{\partial E_0}{\partial x} z \right) d^3x$$ (Similar expression for $\langle J_y \rangle$)

$$\int E_0 \frac{\partial E_0}{\partial x} z \, d^3x = \int \frac{1}{2} \frac{\partial E_0^2}{\partial x} z \, d^3x = \frac{1}{2} \underbrace{\int \frac{\partial E_0^2}{\partial x} dx}_{0} \int z \, dx \, dy$$

$$\int E_0^2 k y \, d^3x = \int E_0^2(\rho) \, k \rho \sin\phi \, \rho \, d\rho \, d\phi \int dz = \int_0^\infty E_0^2(\rho) \rho^2 d\rho \underbrace{\int_0^{2\pi} \sin\phi \, d\phi}_{0} \int dz$$

cylindrically symmetric wave

As a result $\langle J_x \rangle = 0$, and similarly $\langle J_y \rangle = 0$, for a cylindrically symmetric wave. Again, this conforms with the photon picture where a photon moving in the z direction has a spin $\pm\hbar$ in that direction, while $\langle J_x \rangle = \langle J_y \rangle = 0$ for the spin-up or spin-down states along the z axis.